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RITZ METHOD AND SUBSTRUCTURING IN THE STUDY OF VIBRATION WITH STRONG FLUID–STRUCTURE INTERACTION

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The free vibrations of structures coupled with heavy and inviscid fluids are studied considering that there is no cavitation at the interface. Different formulations of the Rayleigh quotient for structures coupled to compressible and incompressible fluids are obtained either considering or neglecting the free surface waves. The Rayleigh–Ritz method is also introduced. It gives a linear eigenvalue problem for an incompressible liquid when the free surface waves are neglected. When the free surface waves are considered, the eigenvalue problem is generally nonlinear both for incompressible and compressible fluids; however, this study proves that a linear eigenvalue problem may be obtained for incompressible fluids and free surface waves included by solving a problem of larger dimension. When the solid coupled with the fluid is a structure modeled with simple components (substructures), it is useful to use the artificial spring method to synthesise substructures. However, this method was never applied to liquid–structure systems before the present study. It was observed that all the substructures in contact with the same fluid volume are not only coupled by the joints, but also by the fluid. An application of the method to a storage tank partially filled with water is also presented.

1. INTRODUCTION

THE STUDY OF STRUCTURAL VIBRATIONS has largely been expanded in recent years as a consequence of (i) increased performance requirements and (ii) the development of more powerful and versatile computers. Research into new approaches to the analysis of complex structures has led to the development of the finite element method (FEM). At the same time, other methods of analysis were also developed. Perhaps the most important class of methods, alternative to the FEM, considers the structure as an assemblage of simple components that are synthesized by using different techniques. The receptance method (Soedel 1993; Huang & Soedel 1993; Amabili 1996b), the artificial spring method (Yuan & Dickinson 1992a,b, 1994; Cheng & Nicolas 1992; Cheng 1994, 1996; Missaoui *et al.* 1996; Amabili 1997), the component mode substitution (Meirovitch 1980) and the transfer matrix method (Yamada *et al.* 1986) belong to this class. However, all these methods were originally developed to study vibration in a vacuum.

A powerful, analytical-numerical technique to study quite simple fluid–structure systems is the Rayleigh–Ritz method. It was introduced in a paper by Zhu (1995), and further elucidated by Amabili (1996c). This method is based on the Rayleigh

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quotient for vibrations of structures coupled to fluids, treated by Zhu (1994). The first part of the present study manipulates the results of Zhu in order to obtain simpler expressions that are more suitable for applications; thus, significant new work is presented for dealing with incompressible fluids.

The Rayleigh–Ritz method is not very efficient in studying structures modeled with more than one component. Thus, when the solid coupled with the fluid is a structure obtained by connecting simple components (substructures), it is a good approach to use a method for synthesizing the substructures. A particularly suitable approach for this purpose is the use of artificial springs, simulating the junctions between the substructures, in the application of the Rayleigh–Ritz method for coupled fluid–structure systems. In the past this method was only used to study structures in a vacuum or a plate-shell system coupled to a resonant cavity (Cheng 1994, 1996). Recently (Amabili 1997), it was applied to model a tank with a flexible bottom resting on a Winkler foundation, partially filled with incompressible liquid, neglecting the free surface waves.

The novelty of the present paper is the generalization of the method applied in the paper of Amabili (1997) to various structures loaded by compressible or incompressible fluids. The fluid can be simulated by neglecting the free surface waves or considering them. Moreover, we show that all the substructures in contact with the same fluid volume are not only coupled by joints but also by the fluid itself. A method to evaluate the coupling energy of the fluid is given.

More specifically, the proposed method gives a linear eigenvalue problem for an incompressible liquid when free surface waves are neglected. In this case, the liquid motion is generated by the vibration of the structure in contact with the liquid and it results in a discernible increase in the kinetic energy of the entire system. In cases when free surface waves and compressible fluids are considered, a nonlinear eigenvalue problem is obtained. This study proves that a linear eigenvalue problem can be obtained for incompressible fluids by increasing the dimension of the mass and stiffness matrices of the system.

In order to verify the potential of the method, we present an application of the technique involving a storage tank filled with an inviscid and incompressible liquid (water) having a free surface normal to the tank axis; this is the application presented in the paper of Amabili (1997), but different results are given. The tank is modeled with a simply supported circular cylindrical shell connected to a simply supported circular plate by an artificial distributed rotational spring of appropriate stiffness.

2. THE RAYLEIGH QUOTIENTS FOR COUPLED FLUID–STRUCTURE SYSTEMS

Undamped normal modes of a thin-walled elastic structure (e.g. a plate or shell) are considered; the equation of motion for this structure, see Figure 1, can be written as

$$\mathbf{N}(\mathbf{u}) = \omega^2 \rho_s h k \mathbf{u}, \tag{1}$$

where $\mathbf{N}$ is a differential operator, $\mathbf{u}$ is the displacement vector of the mean surface of the structure that gives the mode shape, ω is the corresponding circular frequency, ρ_S is the mass density of the material, h is the thickness and k is a parameter which depends on the geometry of the structure. For a shell with a double curvature, $k = A_1 A_2$, where A_1 and A_2 are the radii of the principal curvatures; while for a circular plate, $k = 1$.

For an inviscid, compressible fluid that has an irrotational movement only due to the structural vibration (resting fluid), the deformation potential Φ (not depending on

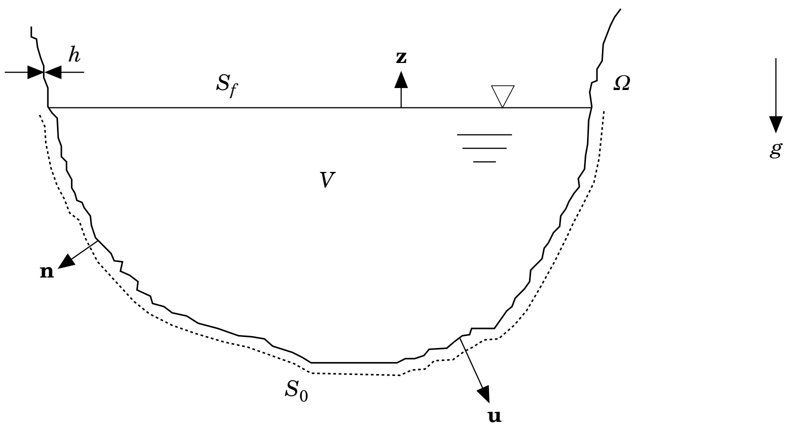

Figure 1. Shell partially in contact with a fluid.

time t) satisfies the Helmholtz equation

$$\nabla^2 \Phi = -\omega^2 c^{-2} \Phi, \tag{2}$$

where c is the velocity of sound in the fluid. The velocity potential $\tilde{\Phi}$ is related to Φ by $\tilde{\Phi} = \mathrm{i}\omega\Phi \mathrm{e}^{\mathrm{i}\omega t}$. If the fluid is incompressible, Φ satisfies the Laplace equation

$$\nabla^2 \Phi = 0. \tag{3}$$

At the fluid-structure interface S_0, the fluid velocity and the wall velocity must be equal; this is the condition of contact between an impermeable wall and a fluid when there is no cavitation at the interface. Therefore, we have

$$\partial \Phi / \partial n = \mathbf{u} \cdot \mathbf{n} \qquad \text{on } S_0, \tag{4}$$

where $\mathbf{n}$ is the unit vector normal to the wall surface and whose positive direction n is outwards in the fluid domain. When the fluid is in contact with a rigid surface S_r, we obtain

$$\partial \Phi / \partial n = 0 \qquad \text{on } S_r. \tag{5}$$

When free surface waves are considered, we have the linearized condition at the fluid free surface S_f (Zhu 1994; Morand & Ohayon 1992)

$$g(\partial \Phi / \partial n) = \omega^2 \Phi \qquad \text{on } S_f, \tag{6}$$

where g is the gravity acceleration and n is the direction orthogonal to the free surface with positive direction outside the fluid volume. When the free surface waves are neglected we impose zero dynamic pressure on S_f (Zhu 1994; Morand & Ohayon 1992);

$$\Phi = 0 \qquad \text{on } S_f. \tag{7}$$

For an unbounded fluid we must impose the radiation condition, i.e. the deformation potential Φ and the velocities of the liquid reach zero when the distance from the solid becomes very large. In fact we require that the velocity of the liquid vanishes at large distances from the structure in such a way that the kinetic energy of the liquid remains finite.

By using the orthogonality relations of wet modes obtained by Huang (1991) and Zhu (1991), we can obtain the Rayleigh quotient for coupled fluid–structure vibrations.

For a compressible fluid and considering the free surface waves, the Rayleigh quotient is given by (Zhu 1994)

$$\omega^2 = \frac{\dfrac{1}{k}\displaystyle\iint_{\Omega} \mathbf{u}\cdot\mathbf{N}(\mathbf{u})\,\mathrm{d}S + \rho_F g \iint_{S_f} \frac{\partial\Phi}{\partial n}\frac{\partial\Phi}{\partial n}\,\mathrm{d}S + \rho_F c^2 \iiint_V \nabla^2\Phi\nabla^2\Phi\,\mathrm{d}V}{\rho_S h \displaystyle\iint_{\Omega} \mathbf{u}\cdot\mathbf{u}\,\mathrm{d}S + \rho_F \iiint_V \boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi\,\mathrm{d}V}, \tag{8}$$

where Ω is the mean surface of the structure, V the fluid volume and ρ_F is the fluid mass density. It is useful to give a mechanical meaning to all the terms of the quotient of equation (8). The numerator gives twice the maximum potential energy of the system; in particular, the first term gives twice the elastic energy of the structure, the second term refers to the free surface waves of the fluid, and the third term gives twice the potential energy stored by the compressible fluid. The denominator of equation (8) expresses twice the reference kinetic energy (maximum kinetic energy divided by ω^2) of the system; the first term refers to the structure and the second to the fluid. When the fluid can be considered incompressible, equation (8) becomes (Zhu 1994)

$$\omega^2 = \frac{\dfrac{1}{k}\displaystyle\iint_{\Omega} \mathbf{u}\cdot\mathbf{N}(\mathbf{u})\,\mathrm{d}S + \rho_F g \iint_{S_f} \frac{\partial\Phi}{\partial n}\frac{\partial\Phi}{\partial n}\,\mathrm{d}S}{\rho_S h \displaystyle\iint_{\Omega} \mathbf{u}\cdot\mathbf{u}\,\mathrm{d}S + \rho_F \iiint_V \boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi\,\mathrm{d}V}. \tag{9}$$

Moreover, since in this case the function Φ is harmonic (in fact it satisfies the Laplace equation (3) when the fluid is incompressible), the expression which gives the reference kinetic energy of the fluid T_F^*,

$$T_F^* = \tfrac{1}{2}\rho_F \iiint_V \nabla\Phi\cdot\nabla\Phi\,\mathrm{d}V, \tag{10}$$

can be simplified into (Amabili 1995)

$$T_F^* = \tfrac{1}{2}\rho_F \iint_{B_f} \Phi\frac{\partial\Phi}{\partial n}\,\mathrm{d}S = \tfrac{1}{2}\rho_F\left(\iint_{S_F} \Phi\frac{\partial\Phi}{\partial n}\,\mathrm{d}S + \iint_{S_0} \Phi\frac{\partial\Phi}{\partial n}\,\mathrm{d}S\right), \tag{11}$$

where B_f is the boundary of the simply-connected fluid domain V. Using equation (5), relation (11) is obtained when the boundary B_f of the fluid volume V is given by $S_0 + S_f + S_r$. The simplification given in equation (11) is a consequence of the application of the Green's theorem to the harmonic function Φ (Lamb 1945). It is also useful to use equation (6) in order to simplify the following expression:

$$\rho_F g \iint_{S_f} \frac{\partial\Phi}{\partial n}\frac{\partial\Phi}{\partial n}\,\mathrm{d}S = \rho_F\omega^2 \iint_{S_f} \Phi\frac{\partial\Phi}{\partial n}\,\mathrm{d}S. \tag{12}$$

By using equations (11) and (12), the Rayleigh quotient can be written as

$$\omega^2 = \frac{\dfrac{1}{k}\displaystyle\iint_{\Omega} \mathbf{u}\cdot\mathbf{N}(\mathbf{u})\,\mathrm{d}S}{\rho_S h \displaystyle\iint_{\Omega} \mathbf{u}\cdot\mathbf{u}\,\mathrm{d}S + \rho_F \iint_{S_0} \Phi\frac{\partial\Phi}{\partial n}\,\mathrm{d}S}. \tag{13}$$

Expression (13) is clearly preferable to equation (9) in applications. Equation (13) is formally unchanged in the case where free surface waves are neglected. Then it is possible to write

$$\omega^2 = \frac{\dfrac{1}{k}\displaystyle\iint_{\Omega} \mathbf{u}\cdot\mathbf{N}(\mathbf{u})\,\mathrm{d}S}{\rho_S h \displaystyle\iint_{\Omega} \mathbf{u}\cdot\mathbf{u}\,\mathrm{d}S + \rho_F \displaystyle\iint_{S_0} \Phi\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S}, \tag{14}$$

where the following further simplification was used

$$T_F^* = \tfrac{1}{2}\rho_F \iint_{S_0} \Phi\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S. \tag{15}$$

It is useful to recall that liquid-filled systems have two families of nodes: the sloshing and the bulging ones (Gupta & Hutchinson 1988). Sloshing modes are caused by the oscillation of the liquid free surface due to the rigid body movement of the container; these modes are also affected by the flexibility of the system. In contrast, the bulging modes are those in which the amplitude of the wall displacement predominates over that of the free surface; in this case, the tank walls and base oscillate with the liquid. Only bulging modes can be studied neglecting free surface waves.

Obviously we must know the mode shape $\mathbf{u}$ or we must give $\mathbf{u}$ *a priori* in order to use equations (8), (13) and (14). The accuracy of the method depends on the accuracy of the choice of the wet mode shapes. In some problems it was proved that the choice of wet mode shapes equal to dry mode shapes gives quite good accuracy. Experiments and discussion on this aspect can be found, for example, in the work of Amabili *et al.* (1995), Amabili & Kwak (1996), Amabili *et al.* (1996) and Amabili (1996a).

3. THE RAYLEIGH–RITZ METHOD FOR COUPLED FLUID–STRUCTURE SYSTEMS

A more accurate solution is obtained by using the Rayleigh–Ritz method (Zhu 1995; Amabili 1996c). The mode shape $\mathbf{u}$ is no longer given *a priori,* but it is expanded in a series by using a finite number of admissible functions $\mathbf{x}_i$, $i = 1, \dots, m$, and appropriate unknown coefficients q_i:

$$\mathbf{u} = \sum_{i=1}^{m} q_i \mathbf{x}_i. \tag{16}$$

The coefficients q_i are computed by the eigenvalue problem that is obtained minimizing the Rayleigh quotient, given by equation (8), with respect to the coefficients q_i. As a consequence of the inclusion principle, the computed eigenvalues approach the actual circular frequencies asymptotically and from above, while the number m of terms considered in the series increases; at the same time the corresponding eigenvectors approach the actual mode shapes.

The deformation potential of the fluid Φ is also described by the eigenvectors, but using the appropriate functions ϕ_i:

$$\Phi = \sum_{i=1}^{m} q_i \phi_i. \tag{17}$$

The functions ϕ_i are unequivocally obtained by the corresponding trial functions $\mathbf{x}_i$ and must satisfy both the Helmholtz equation $\nabla^2\phi_i = -\omega^2 c^{-2}\phi_i$ and the boundary conditions on the free surface S_f of the liquid $g\,\partial\phi_i/\partial n = \omega^2\phi_i$, at the liquid–structure interface $\partial\phi_i/\partial n = \mathbf{x}_i \cdot \mathbf{n}$, and all the other conditions imposed at the fluid boundary. It is clear that both the Helmholtz equation and the free surface condition, when free surface waves are considered, are dependent on the circular frequency ω of the system, which is unknown before the solution of the eigenvalue problem is obtained. Substituting equations (16) and (17) into the Rayleigh quotients, equations (8), (13) and (14), and then minimizing with respect to the coefficients q_i, we obtain

$$\sum_{j=1}^{m} (K_{ij} - \omega_r^2 E_{ij}) q_j = 0, \qquad i = 1, \ldots, m, \tag{18}$$

where $[K_{ij}]$ and $[E_{ij}]$ are the stiffness and mass matrices, respectively, and ω_r is the estimated rth circular frequency of the system. We see that in the Rayleigh quotients we must insert an expression of Φ dependent from ω, and therefore both the matrices $[K_{ij}]$ and $[E_{ij}]$ in equation (18) depend on the circular frequency of the system. The solution of the problem is therefore obtained as a nonlinear eigenvalue problem that must be solved by an iterative algorithm. It could be possible to start the iteration from some assumed values of ω; these values can first be computed, for example, by considering the solution of the problem of an incompressible liquid and neglecting the free surface waves (when we are interested in bulging modes). In fact, in this case, both the fluid deformation potential and boundary conditions are independent of ω, and a linear eigenvalue problem is obtained. In other cases it is useful to start from a very low circular frequency, lower than that of the fundamental mode of sloshing of the fluid considering the structure as rigid. The nonlinear eigenvalue problem can be written as (Schramm & Pilkey 1995)

$$|[K_{ij}(\omega_r^{2(s)})] - \omega_r^{2(s+1)}[E_{ij}(\omega_r^{2(s)})]| = 0, \tag{19}$$

where $\omega_r^{2(s)}$ is the computed rth eigenvalue at step s. This means that for each eigenvalue ω_r^2 a sequence of linear eigenvalue problems must be solved.

3.1. Overcoming the Nonlinear Eigenvalue Problem

When the fluid is incompressible but the free surface waves are retained in the study, it is possible to overcome the nonlinear eigenvalue problem. In fact, the dimension of the problem can be increased utilizing more variables, and a linear eigenvalue problem can be obtained. By using the principle of superposition it is possible to write

$$\Phi = \Phi_B + \Phi_S = \sum_{i=1}^{m} q_i \phi_{B_i} + \sum_{i=1}^{\tilde{m}} h_i \phi_{S_i}, \tag{20}$$

where ϕ_{B_i} and ϕ_{S_i} satisfy the Laplace equation. In particular, the sum Φ_B is the deformation potential obtained neglecting free surface waves, and each term ϕ_{B_i} must satisfy the following boundary conditions:

$$\phi_{B_i} = 0 \quad \text{on } S_f \quad \text{and} \quad \partial\phi_{B_i}/\partial n = \mathbf{x}_i \cdot \mathbf{n} \quad \text{on } S_0. \tag{21a,b}$$

In particular the coefficients q_i in equation (20) are the same as in the Ritz expansion of the mode shape, equation (17). Then the sum Φ_S is the deformation potential due to sloshing, and each term ϕ_{S_i} must satisfy the condition

$$\partial\phi_{S_i}/\partial n = 0 \quad \text{on } S_0. \tag{22}$$

Moreover, the deformation potential must verify the free surface condition

$$g\frac{\partial \Phi}{\partial z} = \omega^2 \Phi \qquad \text{on } S_f, \quad \text{i.e.} \quad g\left[\sum_{i=1}^{\tilde{m}} q_i \frac{\partial \phi_{B_i}}{\partial z} + \sum_{i=1}^{\tilde{m}} h_i \frac{\partial \phi_{S_i}}{\partial z}\right] = \omega^2 \sum_{i=1}^{\tilde{m}} h_i \phi_{S_i} \qquad \text{on } S_f. \quad (23)$$

Equation (23) can be inserted in the eigenvalue problem by increasing its dimension from $m \times m$ to $(m + \tilde{m}) \times (m + \tilde{m})$. Therefore the following linear Galerkin equation is obtained

$$\begin{bmatrix} [K] & [0] \\ [K_1] & [K_2] \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \mathbf{h} \end{Bmatrix} - \omega_r^2 \begin{bmatrix} [M] + [M_a] & [M_S] \\ [0] & [M_1] \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \mathbf{h} \end{Bmatrix} = 0, \qquad (24a)$$

where

$$\boldsymbol{q}^{\mathrm{T}} = \{q_1, \ldots, q_m\} \quad \text{and} \quad \mathbf{h}^{\mathrm{T}} = \{h_1, \ldots, h_{\tilde{m}}\}. \qquad (24b,c)$$

In equation (24) the stiffness matrix $[K]$ and the mass matrix $[M]$ are due to the structure, and the matrix $[M_a]$ is the added mass matrix due to the kinetic energy of the fluid neglecting the free surface waves, i.e. $(T_F^*)_{\text{no wave}} = \frac{1}{2}\rho_F \iint_{S_0} \Phi_B \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}S$; all these matrices have dimension $m \times m$. It is noted that the matrix $[E]$ in equations (18, 19) is given by the sum $[M] + [M_a]$, while $[K]$ is obviously the same. The matrix $[M_S]$ is the added mass matrix associated with the reference kinetic energy due to the sloshing of the fluid; this energy is given by

$$(T_F^*)_{\text{sloshing}} = \frac{1}{2}\rho_F \iint_{S_0} \Phi_S\, \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}S. \qquad (25)$$

Matrices $[K_1]$, $[K_2]$ and $[M_1]$ are due to the vectorial form of equation (23) that is added to the original problem.

It is also interesting to note that, in many problems, the eigenvectors of the *in vacuo* problem can be used as trial functions $\mathbf{x}_i$ in the mode shape expansion; this simplifies the computation of the maximum potential energy of the structure, i.e. the first integral in the numerator of equation (8). This energy can be obtained by multiplying the reference kinetic energy of each eigenvector of the *in vacuo* problem by the corresponding eigenvalue ω_i^2 (the squared circular frequency) of the same problem and by the coefficient q_i, and then adding all the products (Amabili *et al.* 1996; Amabili 1996a); i.e.

$$\frac{1}{k}\iint_{\Omega} \mathbf{x}_i \cdot \mathbf{N}(\mathbf{x}_i)\, \mathrm{d}S = \omega_i^2 \rho_S h \iint_{\Omega} \mathbf{x}_i \cdot \mathbf{x}_i\, \mathrm{d}S, \qquad (26)$$

and then

$$\frac{1}{k}\iint_{\Omega} \mathbf{u} \cdot \mathbf{N}(\mathbf{u})\, \mathrm{d}S = \rho_S h \sum_{i=1}^{m} \omega_i^2 q_i^2 \iint_{\Omega} \mathbf{x}_i \cdot \mathbf{x}_i\, \mathrm{d}S. \qquad (27)$$

In equation (27) we have used the orthogonality of the eigenvectors of the *in vacuo* problem.

4. SUBSTRUCTURE SYNTHESIS

The analysis of a complex structure can be simplified if it is made up of quite simple components (substructures) joined together. In fact, in that case, it is possible to use the knowledge of the dynamic behavior of the substructures to study the whole structure. In particular, the choice of trial functions of each component is simpler than the choice of global trial functions, and equation (27) can be used to evaluate the potential energy of each component. A powerful method to synthesize simple components and study a structure is the artificial spring method.

4.1. The Artificial Spring Method

The artificial spring method is a modification of the classical Rayleigh–Ritz method to synthesize components simplifying the choice of trial functions; it can be attributed to Yuan & Dickinson (1992a) and Cheng & Nicolas (1992).

The Rayleigh–Ritz method was proved to be very efficient in studying complex structures, but in order to obtain correct results the trial functions must satisfy all the geometrical boundary conditions. Even if the extended Rayleigh–Ritz method (Petyt 1971) is utilized, trial functions must satisfy geometrical boundary conditions of the unconstrained structure, and the sum of the series of functions must satisfy the additional constraints. When the Rayleigh–Ritz method is applied to a structure obtained by joining some components together, the boundary conditions require the continuity of translational and rotational displacements between all the rigid junctions of the substructures. This condition gives many problems in the choice of the correct trial functions to use for each single component. The use of artificial springs at the junctions allows us to overcome this difficulty. In particular, the joints between the components of the structure are substituted by translational and rotational artificial springs (see Figure 2) that are distributed along the whole joint length or area. Obviously, each degree of freedom involved in the joint must be simulated by a distributed spring. Then the spring stiffness is chosen to be very high with respect to the structure stiffness, to simulate a rigid junction in numerical computations. The maximum potential energy V_{S_j} stored by the artificial springs simulating the joint j of the structure can be written as

$$V_{S_j} = \frac{1}{2} \int_{l_j} k_j \delta_j \, \mathrm{d}l, \tag{28}$$

where k_j is the stiffness of the translational or rotational artificial springs, δ_j is the relative displacement or rotation between the two components involved in the joint, and l_j is the length or area of the joint j. The total potential energy stored by the artificial springs is obviously the sum of the energies stored in all the joints of the structure, and the double of this energy is included in the numerator of the Rayleigh quotient when the artificial spring method is applied.

The potential energy stored by the springs replaces the continuity condition required

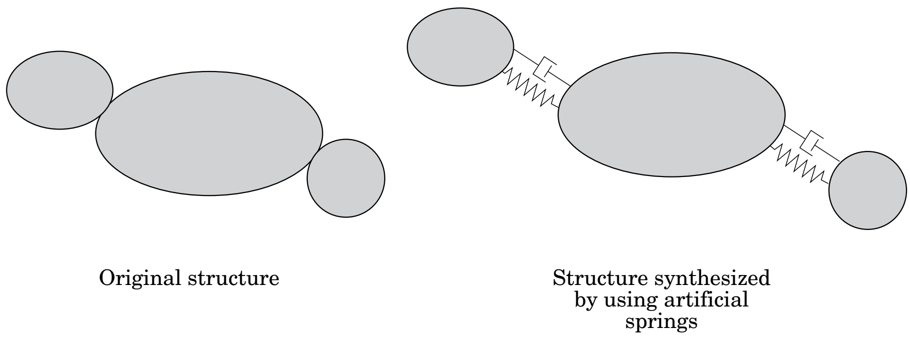

Figure 2. Schematic representation of substructure synthesis.

by the classical Rayleigh–Ritz method if the artificial springs are not introduced. Therefore, the choice of the trial functions is simplified; in particular, one must choose trial functions that allow displacement and rotation of all the springs involved in the junctions and that satisfy the geometrical boundary conditions at the non-connected regions.

4.2. Coupling Due to the Fluid

In Section 4.1 we have described a method for simplifying the analysis of a structure through a technique for synthesizing the components. Now we propose the application of the Rayleigh–Ritz method for coupled fluid–structure systems in the case of substructuring, and in particular by using the artificial spring method.

Let us consider a structure that we choose to divide into l components in the study; we assume that the first p components of the structure are in contact with the simply connected volume V, filled with an incompressible and inviscid fluid. The surface S_0 wetted by the fluid is given by the sum of the contributions of each component; therefore, $S_0 = S_1 + S_2 + \ldots + S_p$, where S_j is the wet surface of component j. The vibration displacement of each component of the structure is $\mathbf{u}_j$, $j = 1, \ldots, l$. By using the admissible functions $\mathbf{x}_{ij}$, $i = 1, \ldots, m$, of the component j we can write

$$\mathbf{u}_j = \sum_{i=1}^{m} q_{ij}\mathbf{x}_{ij}, \tag{29}$$

where $\mathbf{u}_j$ and $\mathbf{x}_{ij}$ are defined in the component j. We associate with any admissible function $\mathbf{x}_{ij}$ the corresponding component ϕ_{ij} of the deformation potential of the fluid, defined in the whole fluid volume V. Each function ϕ_{ij} must satisfy the Laplace equation and all the boundary conditions; in this case the conditions of contact are: $\partial\phi_{ji}/\partial n = \mathbf{x}_{ij} \cdot \mathbf{n}$ on S_j and $\partial\phi_{ij}/\partial n = 0$ on $S_0 - S_j$. Therefore ϕ_{ij} is associated with $\mathbf{x}_{ij}$, considering the structure flexible in S_j and rigid otherwise. The deformation potential Φ is given by

$$\Phi = \sum_{j=1}^{p} \Phi_j, \tag{30}$$

where Φ_j is the contribution to Φ given by the vibration of the component j. Therefore, we have

$$\Phi_j = \sum_{i=1}^{m} q_{ij}\phi_{ij}. \tag{31}$$

The reference kinetic energy of the fluid, equation (15), is now given by

$$T_F^* = \tfrac{1}{2}\rho_F \sum_{i=1}^{p} \iint_{S_i} \sum_{j=1}^{p} \Phi_j \mathbf{u}_i \cdot \mathbf{n}\,\mathrm{d}S. \tag{32}$$

Examining equation (32), we see that there are non-zero contributions also when $i \neq j$. In conclusion, there is a dynamic coupling, due to the fluid, among the components of the structure that are in contact with the same fluid volume, and not only junctions in the structure. Similar relationships are obtained when studying a compressible fluid while retaining the possibility of free surface waves.

5. APPLICATION TO A PLATE-ENDED CIRCULAR CYLINDRICAL TANK

The proposed method is applied to the study of a storage tank filled with an inviscid and incompressible liquid (water) having a free surface normal to the tank axis.

The tank is modeled by a simply supported circular cylindrical shell connected to a simply supported circular plate by an artificial rotational distributed spring (Figure 3) that is assumed to be very rigid. This model is quite realistic because the connection between the plate and the shell gives a reciprocal constraint that can be assumed as a simple support. In many applications the top of the tank is closed by a thin diaphragm or by a ring that constrains the shell displacements similarly to a simple support; only for the purposes of describing correctly beam-like modes of the tank should a free edge be considered as the top.

When a plate is joined to a circular cylindrical shell, in general three displacements and two slope connections could be considered, according to classical thin-shell theory. However, the full treatment of using five connections is not necessary if one investigates only lower modes of the system. For these modes the plate can be assumed inelastic in its plane, and hence to admit only transverse displacements. Moreover, influences of connection deflections in the tangential planes of the shell can be neglected with respect to transverse amplitudes. Therefore, only the radial slope at the plate boundary can be considered coupled to the axial slope of the shell at the bottom end.

In this application only the bulging modes of the structure are investigated and the free surface waves are neglected; the solution is obtained as a linear eigenvalue problem by using an artificial spring in conjunction with the Rayleigh–Ritz method.

A cylindrical polar co-ordinate system $(O; r, \theta, x)$ is introduced, with the origin O at the center of the circular bottom plate. Due to the axial symmetry of the structure, only the modes of the shell and the plate with the same number n of nodal diameters are coupled. Both the axisymmetric vibrations $(n = 0)$ and asymmetric vibrations $(n > 0)$ can be investigated. Besides, it is interesting to note that, due to axial symmetry, for

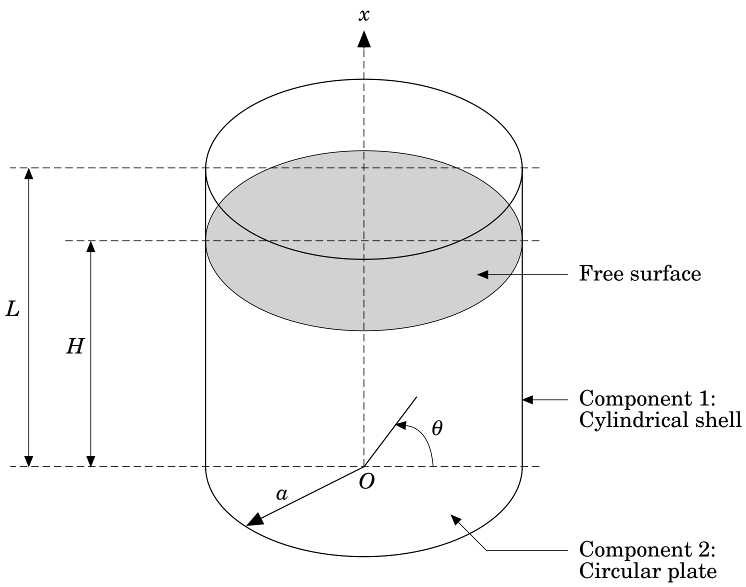

Figure 3. The model used to study the storage tank.

each asymmetric mode there exists a second mode having the same frequency and mode shape, but angularly rotated by $\pi/2n$.

The Rayleigh–Ritz method is applied to find the natural modes of the circular cylindrical tank of radius a and height L. The radial displacement w of the shell wall (Figure 3) can be given by the following expression (Amabili 1997):

$$w(x, \theta) = \cos(n\theta) \sum_{s=1}^{N_1} q_s B_s \sin\left(s\pi \frac{x}{L}\right), \tag{33}$$

where n is the number of nodal diameters, q_S are the Ritz coefficients, B_S are constants depending on the normalization criterion used, and N_1 is the number of terms used in the expansion. The eigenvectors of the empty simply supported shell are used as admissible functions.

The transverse displacement, w_P, of the plate can be given as

$$w_p(r, \theta) = \cos(n\theta) \sum_{i=0}^{N_2} \tilde{q}_i \left[A_{in} \mathrm{J}_n\left(\frac{\lambda_{in} r}{a}\right) + C_{in} \mathrm{I}_n\left(\frac{\lambda_{in} r}{a}\right)\right], \tag{34}$$

where n and i are the number of nodal diameters and circles, respectively, a is the plate radius, $\tilde{q}_i$ are the Ritz coefficients, and λ_{in} is the frequency parameter that is related to the natural frequency of the plate; J_i and I_i are the Bessel function and modified Bessel function of order i, respectively, and N_2 is the number of terms used in the expansion. The mode shapes with an even number n of nodal diameters are symmetric with respect to the longitudinal axis, whereas those with an odd number n are antisymmetric. Mathematical details are given in Amabili (1997).

The numerical solution to the eigenvalue problem is obtained by using the *Mathematica* computer program (Wolfram 1991). Twelve shell modes and twelve plate modes are considered in the Rayleigh–Ritz expansion. The study is addressed to tanks partially filled with water, having $\rho_F = 1000\,\mathrm{kg\,m^{-3}}$. In the case studied, both the shell

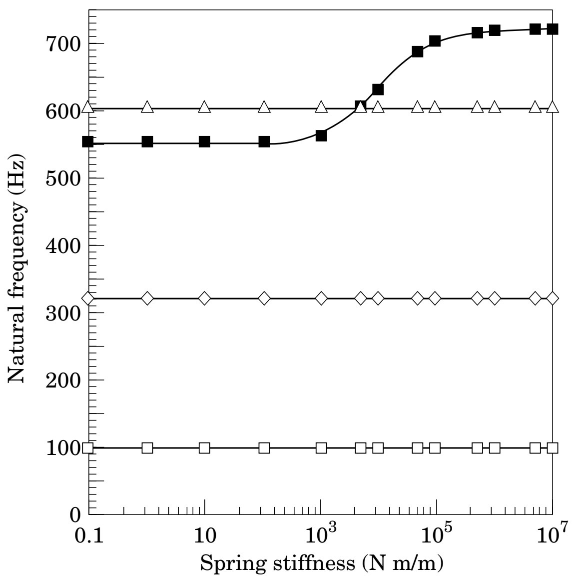

Figure 4. Effect of the spring stiffness on the natural frequencies, in Hz, of the first four modes with four nodal diameters; (□) S1 = first shell-dominant mode; (◇) S2 = second shell-dominant mode; (△) S3 = third shell-dominant mode; (■) P1 = first plate-dominant mode.

and the plate are assumed to be made of steel with the following material properties: Young's modulus $E = 206 \times 10^9$ N m^{-2}, mass density $\rho_S = 7800$ kg m^{-3} and Poisson ratio $\nu = 0{\cdot}3$. The dimensions are: radius $a = 0{\cdot}175$ m, shell length $L = 0{\cdot}6$ m, shell thickness $h_S = 1$ mm and plate thickness $h_P = 2$ mm. The plate and the shell are considered coupled together by a spring with infinite stiffness at the joint. For infinity, one in fact takes a large enough quantity in the calculations. In practice, one sometimes considers a trial value of the spring stiffness and then changes it until one obtains eigenvalues that are not affected by an increment in the stiffness value. However, one can give directly a stiffness value much larger than the plate and shell edge stiffness. In the present case the stiffness value of 10^6 N m/m of the rotational spring connecting plate and shell was used to simulate an infinite stiffness in computations. The effect of the spring stiffness on natural frequencies is shown in Figure 4.

The tank is considered to be completely filled with water to a level of $H = 0{\cdot}6$ m. The first four mode shapes having $n = 2$ nodal diameters are given in Figure 5. Mode shapes are plotted in the tank cross-section defined by $\theta = 0$ and $\theta = \pi$. Obviously, for $n = 2$ we have symmetric mode shapes in this section; the first (Figure 5a) and the third (Figure 5c) modes are shell-dominant (shell displacement larger than plate displacement); the second (Figure 5b) and fourth (Figure 5d) are plate-dominant. Therefore, it

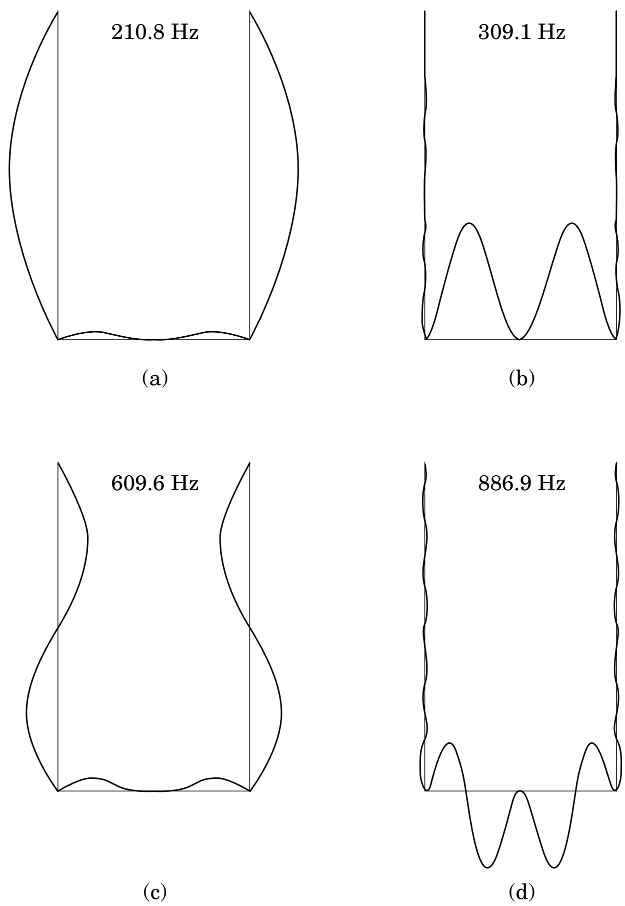

Figure 5. First four modes having $n = 2$ nodal diameters and their natural frequencies. The corresponding *in vacuo* natural frequencies are: (a) 719·9 Hz; (b) 576·1 Hz; (c) 1943·9 Hz; (d) 1400·4 Hz.

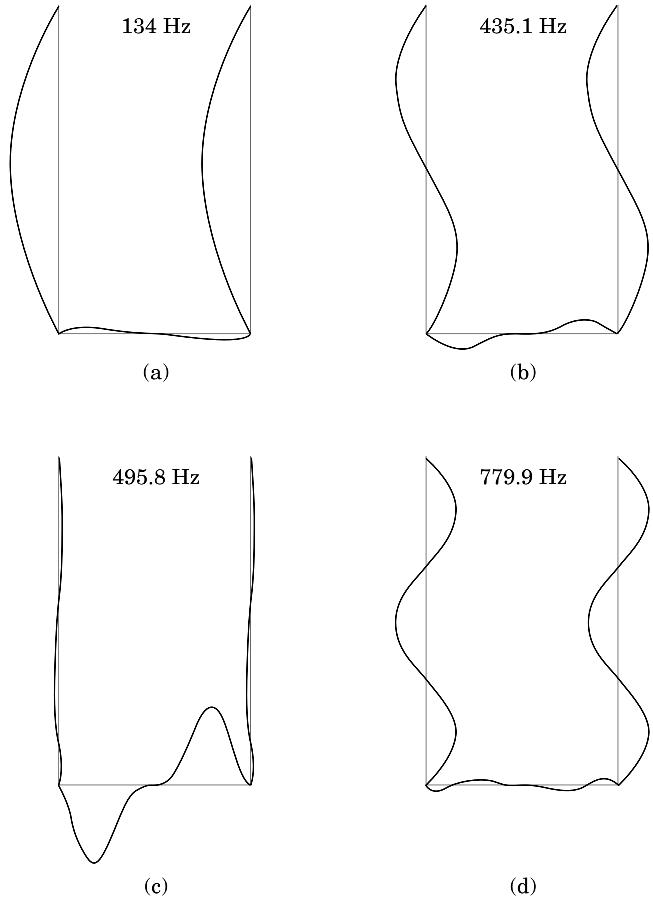

Figure 6. First four modes having $n = 3$ nodal diameters and their natural frequencies. The corresponding *in vacuo* natural frequencies are: (a) 384·5 Hz; (b) 1208·8 Hz; (c) 839·7 Hz; (d) 2062·4 Hz.

is useful to introduce the symbols S1 and S2 to indicate the first and second shell-dominant modes; similarly, P1 and P2 are used for the first and second plate-dominant modes. In the figure are reported also the natural frequencies. The natural frequencies of the corresponding modes in vacuum are given in the caption in order to evaluate the effect of water; obviously, mode shapes are also changed by the presence of water inside the tank. The first four mode shapes having $n = 3$ nodal diameters are presented in Figure 6, while those having $n = 4$ are given in Figure 7; for $n = 3$ we have antisymmetric mode shapes in the cross-section with coordinates $\theta = 0$ and $\theta = \pi$, while for $n = 4$ we have symmetric shapes. Natural frequencies of modes with $n = 4$ nodal diameters are plotted in Figure 8 for different thicknesses of the bottom plate (all the other dimensions are unchanged). Plate-dominant modes are obviously largely affected by the increasing thickness.

Table 1 shows the convergence of the natural frequencies when the number of terms in the expansion is increased. It is interesting to see that the plate-dominant modes display a slower convergence rate in this case. In contrast, very few terms are necessary to assure a good estimation of shell-dominant modes.

Lastly, in order to check the accuracy of the artificial spring method, numerical results obtained by using the present approach were compared to the data presented

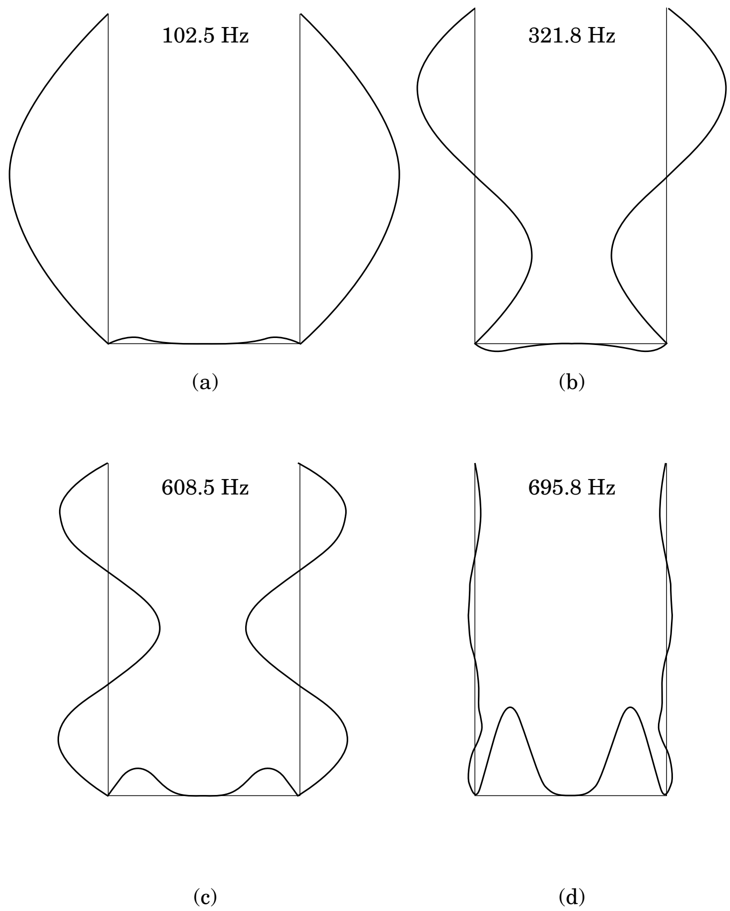

Figure 7. First four modes having $n = 4$ nodal diameters and their natural frequencies. The corresponding *in vacuo* natural frequencies are: (a) 261·7 Hz; (b) 799·6 Hz; (c) 1477 Hz; (d) 1138·8 Hz.

by Huang & Soedel (1993) for an empty plate-ended circular cylindrical shell. Both the shell and the plate are made of steel with the following material properties: $E = 206$ GPa, $\rho_S = \rho_P = 7850\ \text{m}^{-3}$ and $\nu = 0{\cdot}3$. The dimensions are: $a = 0{\cdot}1$ m, $L = 0{\cdot}2$ m, $h_S = h_P = 2$ mm. The comparison is shown in Table 2 for modes having $n = 5$ nodal diameters. A very good agreement between the circualr frequencies given by Huang & Soedel (1993), obtained by using the receptance method, and the present results were found.

6. CONCLUSIONS

The artificial spring method has many advantages in the study of quite complex structures; in fact, admissible functions need not satisfy continuity conditions at the junctions. It was shown that this technique is appropriate for the study of fluid–structure interaction systems, also when the fluid has a free surface. The application to a circular cylindrical tank with a vertical axis, composed of two substructures joined by the relative rotation, tests the effectiveness of the method. Rigid junctions are simulated by springs of infinite stiffness. For infinity, one in fact takes a large enough quantity in the calculations.

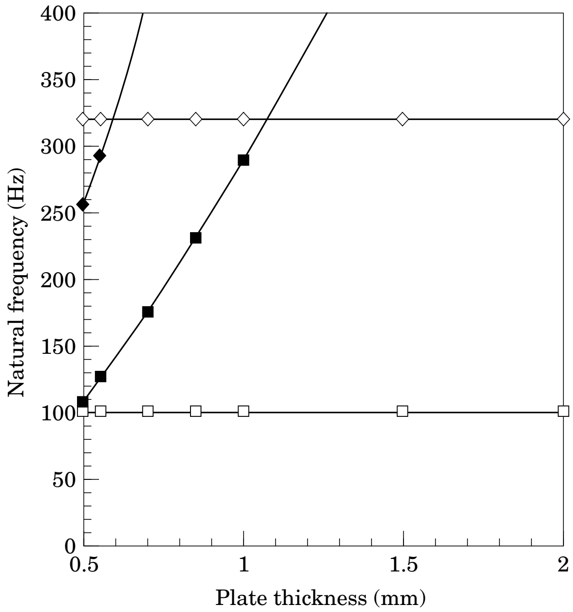

Figure 8. Natural frequencies of the mode shapes with $n = 4$ nodal diameters versus thickness of the bottom plate. (□) S1; (◇) S2; (■) P1; (◆) P2.

TABLE 1

Natural frequencies [Hz] of the first five modes having $n = 4$ nodal diameters of the tank studied, obtained by using a different number $N_1 = N_2$ of terms in the Rayleigh–Ritz expansions of w and w_P

	Natural frequency (Hz)				
$N_1 = N_2$	Mode S1	Mode S2	Mode S3	Mode P1	Mode S4
2	104	323·4	—	875	—
6	103	322·3	608·6	747·7	913·1
10	102·7	322	608·5	717·2	910·7
14	102·5	321·8	608·5	695·8	909·4

TABLE 2

Circular frequencies [rad/s] of the plate-ended circular cylindrical shell studied in the paper of Huang & Soedel (1993); modes with $n = 5$ nodal diameters are considered

Mode	Present study	Huang & Soedel (1993)	Difference (%)
First	9297	9293	0·4
Second	17 695	17 696	0·006
Third	26 296	25 913	1·5
Fourth	28 674	28 328	1·2
Fifth	37 225	37 161	0·2
Sixth	45 127	45 158	0·07

The Rayleigh–Ritz method applied together with artificial springs is modified in order to obtain directly the fluid deformation potential, and is based on the Rayleigh quotient for coupled fluid–structure systems; different expressions of the quotient are given for various applications.

In the case when the effect of the free surface waves is retained in the study, the problem can be reduced to a linear eigenvalue problem by increasing the dimension of the stiffness and mass matrices.

Substructuring applied in conjunction with the Rayleigh–Ritz method and the artificial spring technique is a powerful method for the study of quite complex fluid–structure systems, as an alternative to the more computationally onerous finite element method (FEM) and boundary element method (BEM); moreover, it can also be used as a benchmark for commercial FEM and BEM codes.

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